

EJERCICIOS 4.2

Las respuestas a los problemas impares seleccionados comienzan en la página RESP-13.

En los problemas del 1 al 16, encuentre la serie de Fourier de f en el intervalo dado.

$$1. f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

$$2. f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \leq x < \pi \end{cases}$$

$$3. f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \leq x < 1 \end{cases}$$

$$4. f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \leq x < 1 \end{cases}$$

$$5. f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \leq x < \pi \end{cases}$$

$$6. f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \leq x < \pi \end{cases}$$

$$7. f(x) = x + \pi, \quad -\pi < x < \pi$$

$$8. f(x) = 3 - 2x, \quad -\pi < x < \pi$$

$$9. f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases}$$

$$10. f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \cos x, & 0 \leq x < \pi/2 \end{cases}$$

$$11. f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \\ 0, & 1 \leq x < 2 \end{cases}$$

$$12. f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

$$13. f(x) = \begin{cases} 1, & -5 < x < 0 \\ 1 + x, & 0 \leq x < 5 \end{cases}$$

$$14. f(x) = \begin{cases} 2 + x, & -2 < x < 0 \\ 2, & 0 \leq x < 2 \end{cases}$$

$$15. f(x) = e^x, \quad -\pi < x < \pi$$

$$16. f(x) = \begin{cases} 0, & -\pi < x < 0 \\ e^x - 1, & 0 \leq x < \pi \end{cases}$$

17. Utilice el resultado del problema 5 para demostrar que

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

y

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

18. Utilice el problema 17 para calcular una serie que proporcione el valor numérico de $\pi^2/8$.

19. Utilice el resultado del problema 7 y demuestre que

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

venga del hecho de que las gráficas de las funciones polinómicas las potencias pares de x sean simétricas respecto al eje y , mientras que las potencias impares de x sean simétricas respecto al origen. Por ejemplo,

↓ entero par

$$f(x) = x^2 \text{ es par debido a que } f(-x) = (-x)^2 = x^2$$

↓ entero impar

$$f(x) = x^3 \text{ es impar debido a que } f(-x) = (-x)^3 = -x^3$$

Consulte las figuras 4.4 y 4.5. Las funciones trigonométricas $\cos$ y $\sin$ son pares e impares, respectivamente, ya que $\cos(-x) = \cos x$ y $\sin(-x) = -\sin x$. Las funciones exponenciales $f(x) = e^x$ y $f(x) = e^{-x}$ no son pares ni impares.

Propiedades El teorema siguiente relaciona algunas propiedades de las funciones pares e impares.

TEOREMA 4.2

Propiedades
e impares

- El producto de dos funciones pares es par.
- El producto de dos funciones impares es par.
- El producto de una función par y una impar es impar.
- La suma (resta) de dos funciones pares es par.
- La suma (resta) de dos funciones impares es impar.
- Si f es par, entonces $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
- Si f es impar, entonces $\int_{-a}^a f(x) dx = 0$.

Demostración de b) Supongamos que f y g son funciones pares. Entonces $f(-x) = f(x)$ y $g(-x) = g(x)$. Si definimos el producto $F(x) = f(x)g(x)$, entonces

$$F(-x) = f(-x)g(-x) = (f(x))(g(x)) = F(x)$$

4.1 Orthogonal Functions

25. In R^3 the set $\{\mathbf{i}, \mathbf{j}\}$ is not complete since $\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$. The set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is complete. To see this suppose that $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is orthogonal to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Then

$$0 = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \mathbf{i}) = a(\mathbf{i}, \mathbf{i}) + b(\mathbf{j}, \mathbf{i}) + c(\mathbf{k}, \mathbf{i}) = a(1) + b(0) + c(0) = a.$$

Similarly, $b = 0$ and $c = 0$. Thus, the only vector in R^3 orthogonal to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is $\mathbf{0}$, so $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is complete.

EXERCISES 4.2

Fourier Series

1. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} 1 dx = 1$
 $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi}{\pi} x dx = \frac{1}{\pi} \int_0^{\pi} \cos nx dx = 0$
 $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi}{\pi} x dx = \frac{1}{\pi} \int_0^{\pi} \sin nx dx = \frac{1}{n\pi} (1 - \cos n\pi) = \frac{1}{n\pi} [1 - (-1)^n]$

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx$$
2. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 (-1) dx + \frac{1}{\pi} \int_0^{\pi} 2 dx = 1$
 $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^0 -\cos nx dx + \frac{1}{\pi} \int_0^{\pi} 2 \cos nx dx = 0$
 $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^0 -\sin nx dx + \frac{1}{\pi} \int_0^{\pi} 2 \sin nx dx = \frac{3}{n\pi} [1 - (-1)^n]$

$$f(x) = \frac{1}{2} + \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx$$
3. $a_0 = \int_{-1}^1 f(x) dx = \int_{-1}^0 1 dx + \int_0^1 x dx = \frac{3}{2}$
 $a_n = \int_{-1}^1 f(x) \cos n\pi x dx = \int_{-1}^0 \cos n\pi x dx + \int_0^1 x \cos n\pi x dx = \frac{1}{n^2\pi^2} [(-1)^n - 1]$
 $b_n = \int_{-1}^1 f(x) \sin n\pi x dx = \int_{-1}^0 \sin n\pi x dx + \int_0^1 x \sin n\pi x dx = -\frac{1}{n\pi}$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right]$$
4. $a_0 = \int_{-1}^1 f(x) dx = \int_0^1 x dx = \frac{1}{2}$
 $a_n = \int_{-1}^1 f(x) \cos n\pi x dx = \int_0^1 x \cos n\pi x dx = \frac{1}{n^2\pi^2} [(-1)^n - 1]$
 $b_n = \int_{-1}^1 f(x) \sin n\pi x dx = \int_0^1 x \sin n\pi x dx = \frac{(-1)^{n+1}}{n\pi}$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2 \pi^2} \cos n\pi x + \frac{(-1)^{n+1}}{n\pi} \sin n\pi x \right]$$

$$5. a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{1}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} x^2 \cos nx dx = \frac{1}{\pi} \left(\frac{x^2}{\pi} \sin nx \Big|_0^{\pi} - \frac{2}{n} \int_0^{\pi} x \sin nx dx \right) = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x^2 \sin nx dx = \frac{1}{\pi} \left(-\frac{x^2}{n} \cos nx \Big|_0^{\pi} + \frac{2}{n} \int_0^{\pi} x \cos nx dx \right) = \frac{\pi}{n} (-1)^{n+1} + \frac{2}{n^3 \pi} [(-1)^n - 1]$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2} \cos nx + \left(\frac{\pi}{n} (-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3 \pi} \right) \sin nx \right]$$

$$6. a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 \pi^2 dx + \frac{1}{\pi} \int_0^{\pi} (\pi^2 - x^2) dx = \frac{5}{3} \pi^2$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^0 \pi^2 \cos nx dx + \frac{1}{\pi} \int_0^{\pi} (\pi^2 - x^2) \cos nx dx \\ &= \frac{1}{\pi} \left(\frac{\pi^2 - x^2}{n} \sin nx \Big|_0^{\pi} + \frac{2}{n} \int_0^{\pi} x \sin nx dx \right) = \frac{2}{n^2} (-1)^{n+1} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^0 \pi^2 \sin nx dx + \frac{1}{\pi} \int_0^{\pi} (\pi^2 - x^2) \sin nx dx \\ &= \frac{\pi}{n} [(-1)^n - 1] + \frac{1}{\pi} \left(\frac{x^2 - \pi^2}{n} \cos nx \Big|_0^{\pi} - \frac{2}{n} \int_0^{\pi} x \cos nx dx \right) = \frac{\pi}{n} (-1)^n + \frac{2}{n^3 \pi} [1 - (-1)^n] \end{aligned}$$

$$f(x) = \frac{5\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2}{n^2} (-1)^{n+1} \cos nx + \left(\frac{\pi}{n} (-1)^n + \frac{2[1 - (-1)^n]}{n^3 \pi} \right) \sin nx \right]$$

$$7. a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) dx = 2\pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{n} (-1)^{n+1}$$

$$f(x) = \pi + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

$$8. a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (3 - 2x) dx = 6$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (3 - 2x) \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (3 - 2x) \sin nx dx = \frac{4}{n} (-1)^n$$

$$f(x) = 3 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

$$9. a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} \sin x dx = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} \sin x \cos nx dx = \frac{1}{2\pi} \int_0^{\pi} (\sin(1+n)x + \sin(1-n)x) dx$$

4.2 Fourier Series

$$= \frac{1 + (-1)^n}{\pi(1 - n^2)} \quad \text{for } n = 2, 3, 4, \dots$$

$$a_1 = \frac{1}{2\pi} \int_0^\pi \sin 2x \, dx = 0$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^\pi \sin x \sin nx \, dx \\ &= \frac{1}{2\pi} \int_0^\pi (\cos(1-n)x - \cos(1+n)x) \, dx = 0 \quad \text{for } n = 2, 3, 4, \dots \end{aligned}$$

$$b_1 = \frac{1}{2\pi} \int_0^\pi (1 - \cos 2x) \, dx = \frac{1}{2}$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2} \sin x + \sum_{n=2}^\infty \frac{1 + (-1)^n}{\pi(1 - n^2)} \cos nx$$

$$10. \quad a_0 = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x \, dx = \frac{2}{\pi}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \cos 2nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x \cos 2nx \, dx = \frac{1}{\pi} \int_0^{\pi/2} (\cos(2n-1)x + \cos(2n+1)x) \, dx \\ &= \frac{2(-1)^{n+1}}{\pi(4n^2 - 1)} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \sin 2nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x \sin 2nx \, dx = \frac{1}{\pi} \int_0^{\pi/2} (\sin(2n-1)x + \sin(2n+1)x) \, dx \\ &= \frac{4n}{\pi(4n^2 - 1)} \end{aligned}$$

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^\infty \left[\frac{2(-1)^{n+1}}{\pi(4n^2 - 1)} \cos 2nx + \frac{4n}{\pi(4n^2 - 1)} \sin 2nx \right]$$

$$11. \quad a_0 = \frac{1}{2} \int_{-2}^2 f(x) \, dx = \frac{1}{2} \left(\int_{-1}^0 -2 \, dx + \int_0^1 1 \, dx \right) = -\frac{1}{2}$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi}{2} x \, dx = \frac{1}{2} \left(\int_{-1}^0 (-2) \cos \frac{n\pi}{2} x \, dx + \int_0^1 \cos \frac{n\pi}{2} x \, dx \right) = -\frac{1}{n\pi} \sin \frac{n\pi}{2}$$

$$b_n = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi}{2} x \, dx = \frac{1}{2} \left(\int_{-1}^0 (-2) \sin \frac{n\pi}{2} x \, dx + \int_0^1 \sin \frac{n\pi}{2} x \, dx \right) = \frac{3}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right)$$

$$f(x) = -\frac{1}{4} + \sum_{n=1}^\infty \left[-\frac{1}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} x + \frac{3}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right) \sin \frac{n\pi}{2} x \right]$$

$$12. \quad a_0 = \frac{1}{2} \int_{-2}^2 f(x) \, dx = \frac{1}{2} \left(\int_0^1 x \, dx + \int_1^2 1 \, dx \right) = \frac{3}{4}$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi}{2} x \, dx = \frac{1}{2} \left(\int_0^1 x \cos \frac{n\pi}{2} x \, dx + \int_1^2 \cos \frac{n\pi}{2} x \, dx \right) = \frac{2}{n^2\pi^2} \left(\cos \frac{n\pi}{2} - 1 \right)$$

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi}{2} x \, dx = \frac{1}{2} \left(\int_0^1 x \sin \frac{n\pi}{2} x \, dx + \int_1^2 \sin \frac{n\pi}{2} x \, dx \right) \\ &= \frac{2}{n^2\pi^2} \left(\sin \frac{n\pi}{2} + \frac{n\pi}{2} (-1)^{n+1} \right) \end{aligned}$$

$$f(x) = \frac{3}{8} + \sum_{n=1}^\infty \left[\frac{2}{n^2\pi^2} \left(\cos \frac{n\pi}{2} - 1 \right) \cos \frac{n\pi}{2} x + \frac{2}{n^2\pi^2} \left(\sin \frac{n\pi}{2} + \frac{n\pi}{2} (-1)^{n+1} \right) \sin \frac{n\pi}{2} x \right]$$

13. $a_0 = \frac{1}{5} \int_{-5}^5 f(x) dx = \frac{1}{5} \left(\int_{-5}^0 1 dx + \int_0^5 (1+x) dx \right) = \frac{9}{2}$
 $a_n = \frac{1}{5} \int_{-5}^5 f(x) \cos \frac{n\pi}{5} x dx = \frac{1}{5} \left(\int_{-5}^0 \cos \frac{n\pi}{5} x dx + \int_0^5 (1+x) \cos \frac{n\pi}{5} x dx \right) = \frac{5}{n^2 \pi^2} [(-1)^n - 1]$
 $b_n = \frac{1}{5} \int_{-5}^5 f(x) \sin \frac{n\pi}{5} x dx = \frac{1}{5} \left(\int_{-5}^0 \sin \frac{n\pi}{5} x dx + \int_0^5 (1+x) \sin \frac{n\pi}{5} x dx \right) = \frac{5}{n\pi} (-1)^{n+1}$
 $f(x) = \frac{9}{4} + \sum_{n=1}^{\infty} \left[\frac{5}{n^2 \pi^2} [(-1)^n - 1] \cos \frac{n\pi}{5} x + \frac{5}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{5} x \right]$
14. $a_0 = \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \left(\int_{-2}^0 (2+x) dx + \int_0^2 2 dx \right) = 3$
 $a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi}{2} x dx = \frac{1}{2} \left(\int_{-2}^0 (2+x) \cos \frac{n\pi}{2} x dx + \int_0^2 2 \cos \frac{n\pi}{2} x dx \right) = \frac{2}{n^2 \pi^2} [1 - (-1)^n]$
 $b_n = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi}{2} x dx = \frac{1}{2} \left(\int_{-2}^0 (2+x) \sin \frac{n\pi}{2} x dx + \int_0^2 2 \sin \frac{n\pi}{2} x dx \right) = \frac{2}{n\pi} (-1)^{n+1}$
 $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{n^2 \pi^2} [1 - (-1)^n] \cos \frac{n\pi}{2} x + \frac{2}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{2} x \right]$
15. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} (e^{\pi} - e^{-\pi})$
 $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$
 $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx dx = \frac{(-1)^n n (e^{-\pi} - e^{\pi})}{\pi(1+n^2)}$
 $f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(1+n^2)} \cos nx + \frac{(-1)^n n (e^{-\pi} - e^{\pi})}{\pi(1+n^2)} \sin nx \right]$
16. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} (e^x - 1) dx = \frac{1}{\pi} (e^{\pi} - \pi - 1)$
 $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} (e^x - 1) \cos nx dx = \frac{[e^{\pi} (-1)^n - 1]}{\pi(1+n^2)}$
 $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} (e^x - 1) \sin nx dx = \frac{1}{\pi} \left(\frac{n e^{\pi} (-1)^{n+1}}{1+n^2} + \frac{n}{1+n^2} + \frac{(-1)^n}{n} - \frac{1}{n} \right)$
 $f(x) = \frac{e^{\pi} - \pi - 1}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{e^{\pi} (-1)^n - 1}{\pi(1+n^2)} \cos nx + \left(\frac{n}{1+n^2} [e^{\pi} (-1)^{n+1} + 1] + \frac{(-1)^n - 1}{n} \right) \sin nx \right]$
17. The function in Problem 5 is discontinuous at $x = \pi$, so the corresponding Fourier series converges to $\pi^2/2$ at $x = \pi$. That is,

$$\begin{aligned} \frac{\pi^2}{2} &= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2} \cos n\pi + \left(\frac{\pi}{n} (-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3 \pi} \right) \sin n\pi \right] \\ &= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} (-1)^n = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{6} + 2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) \end{aligned}$$

and

$$\frac{\pi^2}{6} = \frac{1}{2} \left(\frac{\pi^2}{2} - \frac{\pi^2}{6} \right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

4.2 Fourier Series

At $x = 0$ the series converges to 0 and

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} = \frac{\pi^2}{6} + 2 \left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots \right)$$

so

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

18. From Problem 17

$$\frac{\pi^2}{8} = \frac{1}{2} \left(\frac{\pi^2}{6} + \frac{\pi^2}{12} \right) = \frac{1}{2} \left(2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots \right) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

19. The function in Problem 7 is continuous at $x = \pi/2$ so

$$\frac{3\pi}{2} = f\left(\frac{\pi}{2}\right) = \pi + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin \frac{n\pi}{2} = \pi + 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right)$$

and

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

20. The function in Problem 9 is continuous at $x = \pi/2$ so

$$1 = f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1 + (-1)^n}{\pi(1 - n^2)} \cos \frac{n\pi}{2}$$

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{2}{3\pi} - \frac{2}{3 \cdot 5\pi} + \frac{2}{5 \cdot 7\pi} - \cdots$$

and

$$\pi = 1 + \frac{\pi}{2} + \frac{2}{3} - \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} - \cdots$$

or

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots.$$

21. Writing

$$f(x) = \frac{a_0}{2} + a_1 \cos \frac{\pi}{p} x + \cdots + a_n \cos \frac{n\pi}{p} x + \cdots + b_1 \sin \frac{\pi}{p} x + \cdots + b_n \sin \frac{n\pi}{p} x + \cdots$$

we see that $f^2(x)$ consists exclusively of squared terms of the form

$$\frac{a_0^2}{4}, \quad a_n^2 \cos^2 \frac{n\pi}{p} x, \quad b_n^2 \sin^2 \frac{n\pi}{p} x$$

and cross-product terms, with $m \neq n$, of the form

$$a_0 a_n \cos \frac{n\pi}{p} x, \quad a_0 b_n \sin \frac{n\pi}{p} x, \quad 2a_m a_n \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x,$$

$$2a_m b_n \cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x, \quad 2b_m b_n \sin \frac{m\pi}{p} x \sin \frac{n\pi}{p} x.$$

The integral of each cross-product term taken over the interval $(-p, p)$ is zero by orthogonality. For the squared terms we have

$$\frac{a_0^2}{4} \int_{-p}^p dx = \frac{a_0^2 p}{2}, \quad a_n^2 \int_{-p}^p \cos^2 \frac{n\pi}{p} x dx = a_n^2 p, \quad b_n^2 \int_{-p}^p \sin^2 \frac{n\pi}{p} x dx = b_n^2 p.$$

Thus

$$RMS(f) = \sqrt{\frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)}.$$